Temecula Valley Math Competition

Team Round (60 minutes)

April 13, 2024

Names: ____

School: ____

INSTRUCTIONS

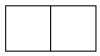
1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. Print your team members' names and school in the spaces provided above.
3. Ask a test proctor if you need any clarifications on a problem.
4. While you can (and should) work with your teammates on the problems, you will submit only one answer sheet for your whole team. DO NOT submit multiple attempts for the same problem.
5. This section consists of 4 problems, each worth 10 points. Put all work towards a solution in the space following the problem statement and the following blank page. If you use extra sheets of paper, write your names and the problem number at the top and attach them to this packet. Draw $\boxed{boxes}$ around your final answers.
6. SCORING: You will receive +10 points for a correct answer OR a maximum of +2 points for work towards an answer. You DO NOT have to write rigorous proofs.
7. HAVE FUN

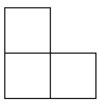
1. Fill in the empty cells of the 5×5 grid with the numbers $1, 2, 3, 4, 5$ such that each row and column contains each number exactly once. Additionally, the numbers must satisfy the greater-than relations indicated by $<$ signs between cells.

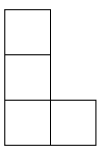
3		5		1		2	$<$	4
4		1		5		3		2
$\vee$								
2		3		4		5		1
5		4		2		1		3
1		2	$<$	3		4		5

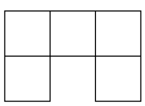
2. Alice and Bob play a game on an infinite square grid given a polyomino **P** (a *polyomino* is a connected subset of squares in the grid).
On each turn, a player claims an empty square and writes their initial in it. Alice wins if she can make a congruent copy of **P** with her squares (i.e. possibly reflected or rotated), while Bob tries to prevent this. Alice moves first.

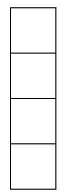
For each **P** below, write *Alice* if Alice can win the game, or *Bob* if Bob can prevent Alice from winning.

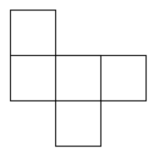
Example:  *Alice*

(a)  *Alice*

(b)  *Alice*

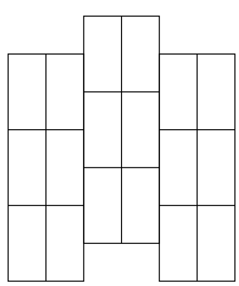
(c)  *Bob*

(d)  *Alice*

(e)  *Bob*

Solution: Alice obviously wins (a). For (b), first note that Alice can get three in a row, at which point there are at least two ways to make an "L" and win. For (d) Alice's strategy is to make a three in a row open on both ends and win. She can do this by threatening four in a row in two different rows.

Bob can block (e) with the following strategy. When Alice claims the square at (x, y), Bob claims $(x - 1, y)$ if x is even, or $(x + 1, y)$ if x is odd. Thus Bob claims at least one square of any copy of **P** that Alice tries to make.
A similar strategy works for (c), but this time Bob always plays in the same domino as Alice according to the tiling pattern below.



3. Let $\triangle ABC$ be a triangle with $\angle A = 60°$, $\angle B = 45°$, $\overline{AC} = 2$, and $\overline{BC} = \sqrt{6}$.
A ball is launched from point D on side AB, bounces off point E on BC, then bounces off point F on AC, and finally returns to D. In other words, the trajectory of the ball forms an inscribed triangle $\triangle DEF$.
The ball bounces such that $\angle ADF = \angle BDE$, $\angle CEF = \angle BED$, $\angle AFD = \angle CFE$.

Determine the values of the following:

(a) $\angle DEF = \boxed{60°}$

(b) $\angle EFD = \boxed{90°}$

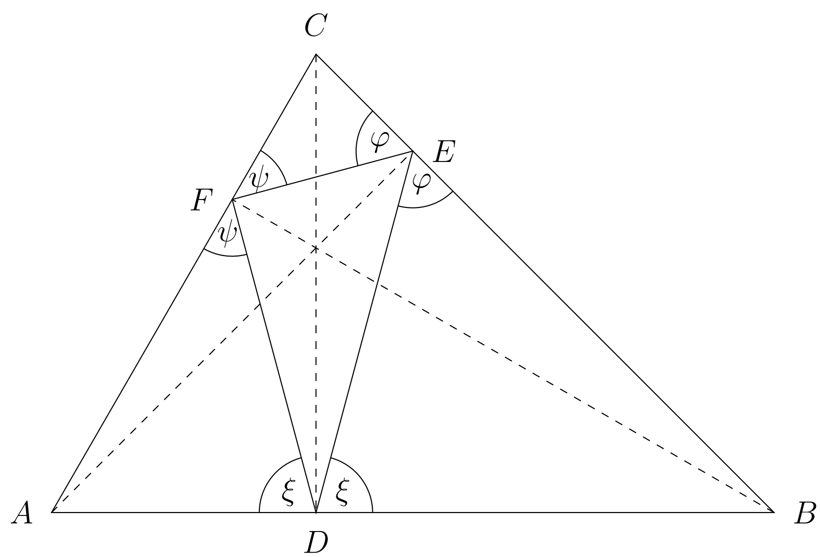
(c) $\angle FDE = \boxed{30°}$

(d) perimeter of $\triangle DEF = \boxed{\sqrt{\frac{3}{2}} + \frac{3\sqrt{2}}{2}}$

Solution: To find the angles we can form a system of three equations

$$75° + \psi + \varphi = 60° + \psi + \xi = 45° + \xi + \varphi = 180°.$$

Solving this gives $\xi = 75°$, $\psi = 45°$, $\varphi = 60°$. Thus $\angle DEF = 60°$, $\angle EFD = 90°$, and $\angle FDE = 30°$.



To find the side lengths of $\triangle DEF$, the key is to realize D,E, and F are the feet of the altitudes of $\triangle ABC$. To see this, note $AFEB$ is a cyclic quadrilateral so $\angle AEB = \angle AFB$. Similarly $\angle AEC = \angle ADC$ and $\angle CFB = \angle CDB$. Since $\angle AEC + \angle AEB = \angle CFB + \angle AFB = \angle ADC + \angle CDB$, all the angles must be right angles.

Now we find $\overline{AD} = \overline{AC}\cos(60°) = 1$ and $\overline{AF} = \overline{AB}\cos(60°) = (1+\sqrt{3})/2$. Using the law of cosines, we find $\overline{DF} = \sqrt{3/2}$. Using that $\triangle DEF$ is a right triangle, we get $\overline{ED} = \sqrt{2}$ and $\overline{EF} = \sqrt{2}/2$. Hence the perimeter is $\sqrt{3/2} + 3\sqrt{2}/2$.

4. Find an $x > 0$ such that $x^2 + 5$, x^2, and $x^2 - 5$ are squares of rational numbers.

Solution: There are infinitely many solutions. One is $x = \boxed{41/12}$. Here is one way to search for a solution. Set $w = \sqrt{x^2 - 5}$ and $y = \sqrt{x^2 + 5}$, so $x^2 - w^2 = y^2 - x^2 = 5$. Rearrange this as

$$\frac{1}{2}(y - w)(y + w) = 5$$
$$(y - w)^2 + (y + w)^2 = (2x)^2.$$

This transformation shows that it is equivalent to find a right triangle with side lengths a, b, c such that the area is 5, via the mapping $(a, b, c) = (y - w, y + w, 2x)$, and $(w, x, y) = ((b - a)/2, c/2, (b + a)/2)$.

Recall that Pythagorean triples can be generated as $(m^2 - n^2, 2mn, m^2 + n^2)$ where $m > n > 0$ and $\gcd(m, n) = 1$. By this or some other method we can list Pythagorean triples

$$\begin{array}{rl} (3,4,5) & (5,12,13) \\ (8,15,17) & (7,24,25) \\ (20,21,29) & (9,40,41). \end{array}$$

Notice that $(9, 40, 41)$ is a right triangle with area $180 = 5 \cdot 6^2$. Hence a required rational triple with area 5 is $(9/6, 40/6, 41/6)$. This corresponds to $(w, x, y) = (31/12, 41/12, 49/12)$.

Remark: $n = 5$ is the smallest positive integer where there exists a sequence of rational squares $x^2 - n, x^2, x^2 + n$. Historically such numbers are called *congruent* numbers.