Temecula Valley Math Competition

Multiple Choice (75 minutes)

March 18, 2023

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the scantron with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive +4 points for each correct answer, −1 point for each problem answered incorrectly, and +0 points for each answer left blank.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. When your proctor gives the signal, begin working on the problems. You will have 75 minutes to complete the test.

1. Nolan is twice as old as Haley. Nolan is 4 years younger than Josele. The average age of Nolan, Haley and Josele is 13. How old is Nolan?

   (A) 7 (B) 10 (C) 14 (D) 17 (E) 21

2. Which of these numbers is smallest?

   (A) 1 (B) $\pi - e$ (C) $45 - \sqrt{2023}$ (D) $\frac{1}{2}$ (E) $289 \cdot 7.007 - 2023$

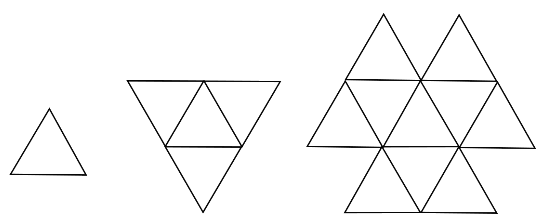
3. Alice forms an arithmetic expression with the numbers $1, 2, 5, 8$ and three operations chosen from $+, -, \div, \times$. Each number is used exactly once, but the same operation can be used multiple times. For example, $(8 \div 2) \times 5 \times 1 = 20$. Which of these *cannot* be the result of evaluating Alice's expression?

   (A) 1 (B) 25 (C) 43 (D) 52 (E) 79

4. Let N be the result of dividing 20212223242526272829 by 5000 and rounding down to the nearest integer. What is the sum of the digits of N?

   (A) 58 (B) 59 (C) 60 (D) 61 (E) 62

5. A new bacteria species *T. equilateralis* forms a colony starting from one triangular bacterium. In the next generation a new bacterium is added on each boundary side. The first three generations are shown below. How many bacteria are in the 23rd generation?



   (A) 829 (B) 760 (C) 729 (D) 694 (E) 680

6. What is the sum of the digits when 2023 is written in base 7?

   (A) 7 (B) 13 (C) 17 (D) 24 (E) 31

7. Suppose ABC is a triangle where $\angle C$ is a right angle and the perimeter is 60. The altitude from vertex C to side AB has length 12. What is the length of the shortest side of the triangle?

   (A) 15 (B) 10 (C) $\frac{25}{2}$ (D) 25 (E) 20

8. Given that x, y, 2 are in a geometric progression, and that x^{-1}, y^{-1}, $9x^{-2}$ are in an arithmetic progression, find the value of xy.

   (A) 3 (B) $\frac{11}{2}$ (C) $\frac{1}{9}$ (D) $\frac{27}{2}$ (E) 15

9. Paul is thinking of three integers. He adds the integers in pairs and gets 65, 186, and 205 respectively. What is the sum of digits of the largest of the integers?

   (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

10. For which value of x does $\log_{2x}(48\sqrt[3]{3}) = \log_{3x}(162\sqrt[3]{2})$?

    (A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 3 (D) 6 (E) $\sqrt{6}$

11. While proving theorems, Bernhard likes to drink two cups of coffee at the same time. Each cup contains 10 ounces of coffee. Whenever he wants a drink, he randomly chooses a cup (with equal probability) and drinks 1 ounce. After a while he picks a cup to drink from and realizes it is empty! What is the probability that there is now exactly 4 ounces left in the other cup?

    (A) $\frac{5}{64}$ (B) $\frac{1}{16}$ (C) $\frac{273}{2048}$ (D) $\frac{3003}{8192}$ (E) $\frac{1001}{8192}$

12. Let C_1 be a circle with radius 2 and center A, and C_2 be a circle with center B. Suppose C_1 and C_2 intersect at two points C and D such that A, B, C, D all lie on a circle of radius 3. What is the radius of C_2?

    (A) $\frac{1}{\sqrt{3}}$ (B) $2\sqrt{3}$ (C) $\sqrt{2}$ (D) 3 (E) $4\sqrt{2}$

13. What is the sum of all three digit positive integers n such that the last three digits of n^2 equals n?

    (A) 376 (B) 1001 (C) 1081 (D) 1021 (E) 1401

14. A collection of integers is called *fourtuitous* if the average of the integers equals 4. How many non-empty collections of positive integers with no more than four elements are fourtuitous? (A collection may contain the same integer multiple times, and the order does not matter).

    (A) 41 (B) 48 (C) 50 (D) 59 (E) 60

15. Sofia is making necklaces, each of which has two Red beads, two Green beads, and two Blue beads. Two necklaces are considered the same if one can be obtained by rotating and/or flipping the other. For example, the necklaces with beads *"RRGGBB"*, *"BBGGRR"*, and *"BGGRRB"* are all considered the same.

    How many distinct necklaces can Sofia make?

    (A) 10 (B) 11 (C) 12 (D) 15 (E) 16

16. The polynomial $x^5 + 20x^4 + 23x^3 + 23x^2 + 20x + 1$ has roots r_1, r_2, r_3, r_4, r_5. Find the value of the sum
$$\sum_{i=1}^{5} r_i^2 + r_i^{-2}.$$

    (A) 354 (B) 569 (C) 612 (D) 708 (E) 1138

17. Define a function by $f(1) = 1$, $f(2n) = f(n)$, and $f(2n+1) = f(n) + 1$ for $n > 1$. What is the greatest integer n such that $f(n) > f(2023)$ and $n < 2023$?

    (A) 2000 (B) 2015 (C) 2018 (D) 2019 (E) 2022

18. A sequence of parentheses is called *balanced* if each opening parenthesis has a matching closing parenthesis. For example, ()() is balanced but (()(is not. How many sequences of 12 parentheses are balanced?

    (A) 42 (B) 89 (C) 132 (D) 256 (E) 332

19. Alan TwoRing invented something he calls the "TwoRing machine". It takes pair of integers (a, b) and, at each step, doubles both numbers.
    However, the machine cannot handle large numbers. Whenever $a \geq 19$, a is replaced by its remainder when divided by 19. Similarly when $b \geq 23$ it is replaced by its remainder when divided by 23.
    Alan starts with $(1, 1)$ and runs his machine for 7^{121} steps. What pair will he get at the end?

    (A) (15, 15) (B) (7,18) (C) (14,13) (D) (1,1) (E) (13,9)

20. Three points are chosen at random on a circle. What is the probability that the triangle formed from the three points is an acute triangle (all of its angles are less than 90°)?

    (A) $\frac{3}{4}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$ (E) $\frac{2}{3}$

21. A sphere with radius 1 rests on top of the xy plane. Its center is at xyz coordinates $(0,0,1)$. A light source is at coordinates $(1,0,4)$, so that the sphere casts a shadow on the xy plane. What is the area of the shadowed region?

(A) $\frac{3\pi\sqrt{2}}{2}$ (B) $\frac{6\pi\sqrt{3}}{5}$ (C) $\frac{\pi\sqrt{3}}{4}$ (D) $\frac{\pi}{3}$ (E) $\frac{\pi\sqrt{2}}{2}$

22. Determine the number of positive integers $n \leq 1000$ such that the sum of the digits of $5n$ and n are the same.

(A) 75 (B) 81 (C) 85 (D) 89 (E) 91

23. Consider the integer

$$N = \sum_{k=0}^{20} (-1)^k \binom{20}{k} (23-k)^{20}.$$

What is the largest power of 2 that divides N?

(A) 2^5 (B) 2^{11} (C) 2^{18} (D) 2^{24} (E) 2^{31}

24. Let $p(x) = x^4 + ax^3 + bx^2 + cx + 1$ where $a, b, c \geq 0$ are non-negative real numbers. If $p(x) = 0$ has four real solutions, what is the minimum possible value of $P(2)$?

(A) 3 (B) $\sqrt[3]{81}$ (C) 27 (D) 81 (E) $\sqrt[3]{4}$

25. Suppose $I, \heartsuit, T, V, M, C$ are positive numbers that satisfy the system of equations

$$\begin{aligned} I(I+\heartsuit+T+V+M+C) = 1 = (I+\heartsuit+T+V+M+C)C \\ (I+\heartsuit)(\heartsuit+T+V+M+C) = 1 = (I+\heartsuit+T+V+M)(M+C) \\ (I+\heartsuit+T)(T+V+M+C) = 1 = (I+\heartsuit+T+V)(V+M+C). \end{aligned}$$

What is the value of MT?

(A) $2\sqrt{2}+1$ (B) $\frac{2\sqrt{2}-3}{2}$ (C) $5\sqrt{2}+7$ (D) $\frac{5\sqrt{2}-7}{2}$ (E) $\frac{\sqrt{5}+1}{6}$