Temecula Valley Math Competition

Team Round (60 minutes)

March 18, 2023

Names: ____

School: ____

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. Print your team members' names and school in the spaces provided above.
3. While you can (and should) work with your teammates on the problems, you will submit only one answer sheet for your whole team. DO NOT submit multiple attempts for the same problem.
4. This section consists of 4 problems, each worth 10 points. Put all work towards a solution in the space following the problem statement and the following blank page. If you use extra sheets of paper, write your names and the problem number at the top and attach them to this packet.
5. SCORING: You are graded based on the correctness, completeness, and clarity of your solutions. Excepting numeric answers, all answers must be rigorously justified. Clearly state any theorems that you use.

1. In jigsaw sudoku, the goal is to place the digits 1 to 9 in a 9×9 grid so that each digit appears *exactly once* in each row, column, and jigsaw piece (the jigsaw pieces are the irregular pieces outlined in black). Solve the jigsaw sudoku below.

10 pt for solving (the solution is unique).

3	1	2	5	8	6	7	4	9
9	4	1	8	3	7	2	5	6
2	6	5	9	4	8	3	1	7
7	3	4	2	6	9	1	8	5
1	8	9	7	5	3	6	2	4
6	9	7	4	2	5	8	3	1
5	2	3	6	1	4	9	7	8
8	5	6	1	7	2	4	9	3
4	7	8	3	9	1	5	6	2

2. Suppose that a triangle has side lengths a, b, c and area A. Prove the following inequality, and find the conditions for which equality is achieved.

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}A.$$

1 pt for showing equality is achieved for equilateral triangle.
9 pt for proof of inequality. Partial credit may be awarded for progress on a proof.

Solution:
There are many ways to prove this, here is one.

$$\begin{gathered}(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0 \\ a^2 + b^2 + c^2 \geq ab + bc + ca.\end{gathered}$$

Clearly, equality holds in the inequality above if and only if the triangle is equilateral. Denote by α, β, and γ the angles of the triangle such that

$$2A = ab\sin\alpha = bc\sin\beta = ca\sin\gamma.$$

Then

$$ab + bc + ca = 2A\left(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma}\right).$$

In the case of equality the the left hand side equals $2A \cdot 6/\sqrt{3} = 4\sqrt{3}A$. Hence $a^2 + b^2 + c^2 \geq 4\sqrt{3}A$. $\square$

Another way: by Heron's formula and AM-GM

$$\begin{aligned}16A^2 &= (a+b+c)(b+c-a)(a+c-b)(a+b-c) \\ &\leq (a+b+c)\left(\frac{a+b+c}{3}\right)^3\end{aligned}$$

where equality occurs if and only if $a = b = c$ (by AM-GM). Taking square roots we get

$$\begin{aligned}4A &\leq \frac{1}{3\sqrt{3}}(a+b+c)^2 \\ &\leq \frac{1}{3\sqrt{3}} \cdot 3(a^2+b^2+c^2) = \frac{1}{\sqrt{3}}(a^2+b^2+c^2).\end{aligned}$$

where in the last line we used $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ by Cauchy-Schwarz. Hence $4A\sqrt{3} \leq a^2 + b^2 + c^2$. $\square$.

3. We say an integer has a *hexary* expansion if it can be represented as a sum of terms of the form $2^n \cdot 3^m$, $n, m \geq 0$ such that for every pair of terms, one term is not a divisor of the other. For example, $19 = 2^2 + 2^1 3^1 + 3^2$ is a valid hexary expansion.

   Prove or disprove: Every positive integer has a hexary expansion.

   1 pt for claiming the statement is true.
   9 pt for a complete proof.
   For an induction proof, deduct **1 pt** for no base case. Partial credit may be awarded for progress on a proof.

   Solution:

   We claim every positive integer has a hexary expansion. We prove this by induction. Clear it is true for $1 = 1$, $2 = 2^1$, $3 = 3^1$. Now suppose every integer $k \leq n-1$ has a hexary expansion for $n > 4$.

   If n is even, then by induction $n/2$ has a hexary expansion, say $a_0 + a_1 + \cdots + a_m$. Then $n = 2a_0 + 2a_1 + \cdots + 2a_m$ is a valid hexary expansion since $2a_i | 2a_j \iff a_i | a_j$.

   If n is odd, let 3^s be the largest power of 3 such that $3^s \leq n < 3^{s+1}$, $s \geq 1$. If n is a power of 3 then we're done, so we can assume $n - 3^s > 0$. By induction we can write a hexary expansion $(n - 3^s)/2 = a_0 + a_1 + \cdots + a_m$.
   We claim $n = 2a_0 + \cdots + 2a_m + 3^s$ is a hexary expansion for n. Clearly, $2a_i \nmid 3^s$, $0 \leq i \leq m$. Moreover, $2a_i < n - 3^s < 3^{s+1} - 3^s = 2 \cdot 3^s$ so $a_i < 3^s$. So $3^s \nmid a_i$ and therefore $3^s \nmid 2a_i$. This proves the claim and completes the proof. □

4. (a) n^2 cells are arranged in a $n \times n$ square grid. Initially, some cells are infected and the rest are healthy. After one minute, every healthy cell with at least two infected neighbors (horizontally or vertically) becomes infected. Infected cells stay infected. This process continues until the infection cannot reach more cells. Find, with proof, the minimum number of initially infected cells that guarantees all cells will eventually become infected.

   (b) Now consider n^3 cells in a $n \times n \times n$ cube. The infection spreads the same way, but now cells can have neighbors in six directions. Find the minimum number of initial infected cells that guarantees all cells will eventually become infected.

for both (a) and (b):
1 pt for getting the minimum number and showing it suffices
4 pt for complete proof that the minimum number is necessary

Solution:
For a $n \times n$ grid the minimum number is $\boxed{n}$. There are many initial configurations that show this number suffices, for example the n diagonal cells. To prove that n is necessary, we view the cells as squares. Let S be a set of initially infected squares. The number of edges on the boundary of S cannot increase when S infects a new square, as the newly infected square shares at least 2 edges with the boundary of S. Hence the inital boundary must be at least $4n$, the boundary of the whole grid. The number of boundary edges is maximized S is union of squares that are not neighbors. We conclude that $|S| \geq n$ in order for the whole grid to be infected.

For a $n \times n \times n$ cube the minimum number is $\boxed{\lceil 3n/2 \rceil}$. In what follows we will index the cells by integer coordinates (x, y, z), where $0 < x, y, z \leq n$. We can construct an initial set of infected cells S like so: S contains the diagonal cells of the $z = 1$ grid: $(1, 1, 1), (2, 2, 1), \ldots (n, n, 1)$, in addition to every other cell of the $x = y = 1$ row,

- $(1, 1, 1), (1, 1, 3) \ldots, (1, 1, \frac{n}{2}), (1, 1, \frac{n}{2} + 1) \ldots, (1, 1, n)$ if n is even,
- $(1, 1, 1), (1, 1, 3) \ldots, (1, 1, \frac{n+1}{2}) \ldots, (1, 1, n)$ if n is odd.

S contains exactly $\lceil 3n/2 \rceil$ cells. It's easy to see that the $x = y = 1$ row and the $z = 1$ grid will eventually be completely infected, and then infect the whole cube.

To show that $\lceil 3n/2 \rceil$ cells are necessary, we use another invariant. If B is a box of cells, define $M(B)$ to be the sum of its side lengths minus 1, i.e.

$$M([a_1, a_2) \times [b_1, b_2) \times [c_1, c_2)) = (a_2 - a_1) + (b_2 - b_1) + (c_2 - c_1) - 1.$$

Note that for a union of boxes we have the sharp inequality $M(B_1 \cup B_2) \leq M(B_1) + M(B_2)$. We extend M to a function on arbitrary set of cells S by defining $M(S)$ to be the minimum of the sum $\sum_i B_i$ over all possible ways to write S as the union of boxes B_i.

Also, for two boxes B_1, B_2 denote by $B_1 \vee B_2$ the minimum box that contains them both. Note that if B_1 and B_2 are infected and the distance between them is less than 3, then $B_1 \vee B_2$ will be infected. Morever $M(B_1 \vee B_2) \leq M(B_1) + M(B_2)$.

With this preparation we can prove the original claim. Let B be the $n \times n \times n$ cube and let $S \subseteq B$ be a set that can infect B. Write $S = B_1 \cup B_2 \cup \cdots \cup B_m$ as the union of boxes such that $M(S)$ is minimal. If $m = 1$ then $S = B$ and we are done, so assume $m \geq 2$. There must be some two boxes B_i, B_j that are distance at most than 2 apart, otherwise S cannot infect more cells. Let $S_1 = S \cup (B_i \vee B_j)$. Then

$$M(S) \geq M(B_i \vee B_j) + \sum_{k \neq i,j} B_k \geq M(S_1).$$

Continuing in this way, we obtain $S \subset S_1 \subseteq S_2 \subseteq \cdots \subseteq B$, such that $M(S) \geq M(S_1) \geq M(S_2) \geq \cdots \geq M(B)$. Since $M(S) \leq 2|S|$ and $M(B) = 3n - 1$, we conclude that $2|S| \geq 3n - 1$, so $|S| \geq \lceil 3n/2 \rceil$. $\square$