TVMC

2022 High School Math Competition

Multiple Choice Test

February 12, 2022

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the scantron with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive $+4$ points for each correct answer, -1 point for each problem answered incorrectly, and $+0$ points for each answer left blank.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have 75 minutes to complete the test.

1. Find the last digit of
$$2022^2 - 2021^2 + 2020^2 - 2019^2 + 2018^2 - \cdots + 2^2 - 1^2.$$
(A) 0 (B) 2 (C) 3 (D) 5 (E) 7

2. Evaluate
$$\frac{2+4+6+\cdots+2022}{1+3+5+\cdots+2021}.$$
(A) $\frac{1012}{1011}$ (B) 1011 (C) $\frac{2022}{2021}$ (D) $\frac{1010}{1012}$ (E) $\frac{1013}{1012}$

3. A 6-sided loaded die labeled with sides labeled 1 through 6 lands on an even number with probability p and lands on an odd number twice as often as it lands on even. Compute the probability of rolling a 5 as a fraction in lowest terms.
(A) 1/2 (B) 1/3 (C) 1/6 (D) 1/9 (E) 2/9

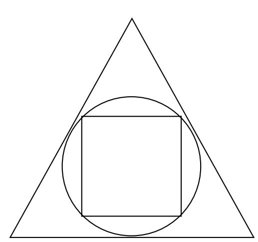
4. Pick a divisor d of 2022 uniformly at random (including 1 and 2022). What's the probability that d contains the digit 1?
(A) 1/2 (B) 1/3 (C) 1/4 (D) 1/8 (E) 1/12

5. Given that $\log_2 10 \approx 3.32$, how many digits does 20^{22} have when written in binary?
(A) 99 (B) 96 (C) 94 (D) 29 (E) 28

6. A pair of twin primes (p, q) is a collection of two primes p and q such that $|p - q| = 2$. The sum of a pair of twin primes is $p + q$. Find the sum of the sum of all pairs of twin primes with $p < 25$. Note that (p, q) and (q, p) is the same pair of twin primes and should only be counted once.
(A) 44 (B) 60 (C) 75 (D) 80 (E) 100

7. Find the value of $30 + \sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}$.
(A) 45 (B) 35 (C) 25 (D) 36 (E) 49

8. A unit square is inscribed in a circle which is inscribed in an equilateral triangle, as shown in the diagram. What is the total perimeter of the three shapes?



(A) $4+\pi+\frac{\sqrt{3}}{2}$ (B) $4+\sqrt{2}\pi+3\sqrt{6}$ (C) $4+\sqrt{2}\pi+3\sqrt{3}$ (D) $8+\sqrt{2}\pi+\sqrt{6}$ (E) $4\sqrt{3}+2\pi$

9. Suppose we make a $3 \times 3 \times 3$ cube made up of red unit cubes and dunk the large cube in a vat of blue paint. After this, we remove one $1 \times 1 \times 1$ corner of the cube as well as a $1 \times 1 \times 1$ piece adjacent to it, revealing some red on the inside. We discard the removed cubes. If we randomly select one of the remaining cubes, what's the probability the cube has a visible red face? By adjacent, we mean that the small cubes share a face.

   (A) 4/5 (B) 1/5 (C) 49/54 (D) 5/54 (E) 2/27

10. If $x + y = xy = 3$, find the value of $x^3 + y^3$.

    (A) -27 (B) -9 (C) 0 (D) 3 (E) 27

11. Evaluate

$$\left[\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}\right]^{1/n}.$$

    (A) 1 (B) 2 (C) $2^{1-1/n}$ (D) n (E) n^2

12. Kasey has a bag of m&m candies which contains 5 green, 3 blue, and 9 red candies. He then chooses two candies from the bag and eats them. The probability that the two candies he ate are the same color can be expressed as a fraction a/b with $\gcd(a, b) = 1$. Find $a + b$.

    (A) 11 (B) 167 (C) 205 (D) 301 (E) 410

13. How many integer solutions (a, b) does the equation $ab - 3a = 7 + 2b$ have?

    (A) 0 (B) 1 (C) 2 (D) 4 (E) 8

14. There are three primes that necessarily divide every 6-digit number of the form $xyzxyz$. Find the sum of these primes.

    (A) 8 (B) 15 (C) 23 (D) 31 (E) 41

15. Define the operation $\oplus$ on the integers by $x \oplus y = xy + x + y$. Compute

$$(\cdots(((1 \oplus 2) \oplus 3) \oplus 4) \cdots \oplus 99).$$

    (A) $98! + 98$ (B) $99! - 1$ (C) $99! + 99$ (D) $100! - 1$ (E) $100! + 1$

16. Three positive integers lie in an arithmetic progression with common difference d. The middle of these three numbers has exactly three divisors. Given that 7 divides each of the three numbers, how many possible values of d are there?

    (A) 0 (B) 1 (C) 6 (D) 7 (E) infinitely many

17. Find the smallest prime p such that p divides $(p - 2)^{p-2} - (p - 3)^{p-3}$.

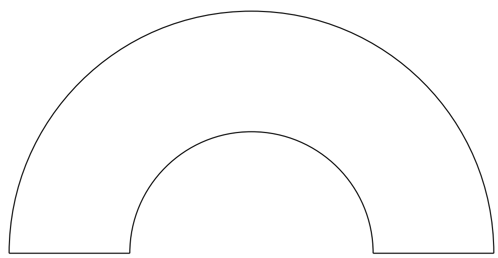
    (A) 5 (B) 7 (C) 11 (D) 13 (E) 17

18. Suppose that A is the largest possible area of a triangle with integer side lengths and perimeter 14. What is A^2?
(A) 144 (B) 84 (C) 336 (D) 196 (E) 368

19. Let $p(x) = ax^3 + bx^2 + cx + 2a$ be such that $p(1 - i) = 0$, where $a, b, c \in \mathbb{R}$ with $a \neq 0$. Which of the following is a root of $p(x)$?
(A) -1 (B) 0 (C) 1/2 (D) 1 (E) 2

20. Find the number of positive integers n such that
$$1^n + 2^n + 3^n + \cdots + 9^n$$
is a prime number.
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

21. The United States Senate floor has seats arranged in a region defined by the difference of two semicircles with the same centers as shown in the diagram. Suppose the longest straight line path contained entirely within the region has length 32 meters. Find the area of the region in square meters.



(A) 32π (B) 96π (C) 108π (D) 128π (E) 256π

22. Suppose that we take a unit circle and inscribe a regular polygon with 2^n sides within it. Let A_n denote the area of the inscribed polygon. Compute
$$\prod_{k=2}^{\infty} \frac{A_k}{A_{k+1}}.$$
(A) $1/\pi$ (B) $2/\pi$ (C) $\pi^2/6$ (D) π (E) 1/2

23. How many rational points are there on the ellipse
$$2x^2 + 5y^2 = 1?$$
A rational point (x, y) is a pair of rational numbers that satisfies the equation.
(A) 0 (B) 1 (C) 2 (D) 100 (E) infinitely many

24. Ada has a spherical tangerine and a sharpie. She draws five black dots on the surface of the tangerine. What is the probability that there exists a set of four dots that all lie in the same hemisphere?
(A) 1/4 (B) 4/5 (C) 2/3 (D) 1/2 (E) 1

25. At first, a room contains 12 people. After each minute, either 1 person enters the room or 4 people leave. After 1728 minutes, which of the following could be the number of people in the room?

    (A) 744 (B) 743 (C) 742 (D) 741 (E) 740