# 2022 High School Math Competition

Free Response Test

February 12, 2022

**Name:** ______________________________

**School:** ______________________ **ID:** ______________

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. Print your name, school, and school ID number in the spaces provided above.
3. This section consists of 4 problems, each worth 10 points. Put all work towards a solution in the space following the problem statement and the following blank page. If you use extra sheets of paper, write your name and the problem number at the top and attach them to this packet.
4. SCORING: You are graded based on the correctness, completeness, and clarity of your solutions. Excepting numeric answers, all answers must be rigorously justified. Clearly state any theorems that you use.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. When your proctor gives the signal, begin working on the problems. You will have 60 minutes to complete the test.

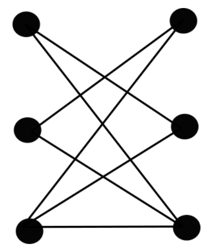
1. (10 pts) Completed square

| | | | | |
|---|---|---|---|---|
| 25 | 16 | 3 | 9 | 12 |
| 8 | 14 | 22 | 20 | 1 |
| 17 | 5 | 6 | 13 | 24 |
| 11 | 23 | 19 | 2 | 10 |
| 4 | 7 | 15 | 21 | 18 |

2. (a) (2 pts) $\binom{6}{3} = \boxed{20}$

   (b) (3 pts) The max is $\boxed{9}$, achieved by the complete bipartite graph shown below.

   (c) (5 pts) The max is $\boxed{\lfloor n^2/4 \rfloor}$. When $n = 2m$, the maximum is achieved by the complete bipartite graph $K_{m,m}$, i.e. two groups of $m$ students where each student is a pair with all $m$ students of the other group. Similarly, when $n = 2m + 1$ is odd it is achieved by $K_{m,m+1}$.



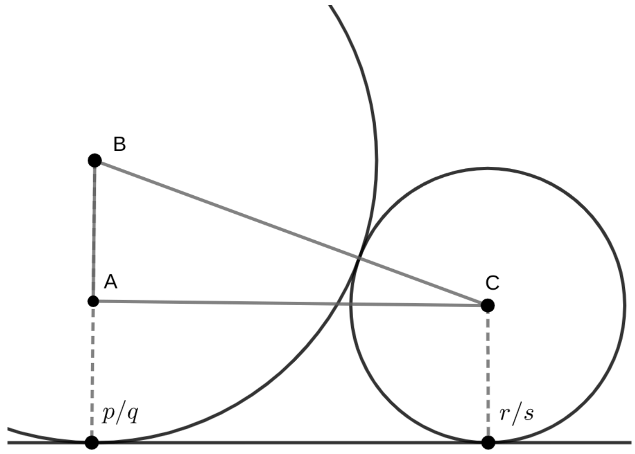
3. (a) (2 pts) $L(16) = L(32) = L(64) = \boxed{1}$

   (b) (3 pts) $L(2^k) = \boxed{1}$. After half the cookies are eaten, we can relabel all the cookies $m \to 2m - 1$ to get the same starting point as $L(2^{k-1})$. Thus $L(2^k) = L(2^{k-1})$ for all $k > 0$.

   (c) (5 pts) Using the same reasoning as before (relabeling $m \to 2m+1$), we see $L(2m+1) = 2L(m)+1$. Let $a_k = 2^0 + 2^1 + \cdots + 2^k$, then

$$L(a_k) = 2L(a_{k-1}) + 1$$

   Since $a_k$ and $L(a_k)$ satisfy the same recurrence, and $a_1 = 3 = L(a_1)$, we conclude that $\boxed{L(n) = n}$.




4. (a) (7 pts) Without loss of generality, assume $p/q < r/s$. Further, assume $q < s$ (the other case is identical). If $C(p,q)$ and $C(r,s)$ are tangent, then in the diagram above we have

$$AC = \frac{r}{s} - \frac{p}{q}$$
$$AB = \frac{1}{2q^2} - \frac{1}{2s^2}$$
$$BC = \frac{1}{2q^2} + \frac{1}{2s^2}.$$

Expand out $AB^2 + AC^2 = BC^2$ and multiply through by $q^4s^4$ to obtain $1 = (qr - ps)^2$. Hence $|qr - ps| = 1$. Conversely if $|qr - ps| = 1$, then we obtain the same equation for $AB, AC$, and $BC$. In particular, $BC$ is the sum of the radii so $C(p,q)$ and $C(r,s)$ are tangent.

(b) (3 pts) Using the previous part, we have to show $|(q+s)p-(r+p)q| = 1$ and $|(q+s)r-(r+p)s| = 1$. Both of these follow from $|qr - ps| = 1$.