2022 High School Math Competition

Free Response Test

February 12, 2022

Name: ____

School: ______________________ **ID:** ____________________

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. Print your name, school, and school ID number in the spaces provided above.
3. This section consists of 4 problems, each worth 10 points. Put all work towards a solution in the space following the problem statement and the following blank page. If you use extra sheets of paper, write your name and the problem number at the top and attach them to this packet.
4. SCORING: You are graded based on the correctness, completeness, and clarity of your solutions. Excepting numeric answers, all answers must be rigorously justified. Clearly state any theorems that you use.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. When your proctor gives the signal, begin working on the problems. You will have 60 minutes to complete the test.

1. A magic square is a square grid of integers where the sum of every column and every row is the same. Complete the magic square below.

		3		
8			20	
	5		13	
11	23	19		
	7	15	21	18

2. Alice, Bob, Clare, Dmitri, Eddy, and Fiona go to the first meeting of their school's math club. At first, nobody knows each other, but eventually certain pairs of people meet. Call two people a *pair* if they have met, and call three people a *triangle* if all of them have met each other.

   (a) How many triangles are formed if everyone meets each other?

   (b) What is the maximum number of pairs possible where there is *no* triangle?

   (c) Suppose now there are $n > 6$ people going to the meeting. What is the maximum number pairs possible where there is *no* triangle, in terms of n? (*Hint*: split the people into two groups)

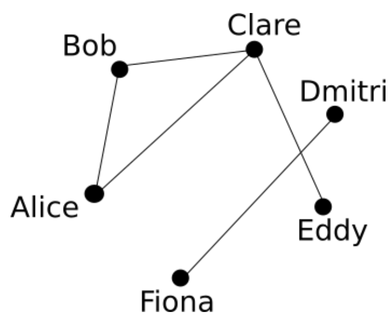


Figure 1: Example with 5 pairs and 1 triangle (lines represent pairs)

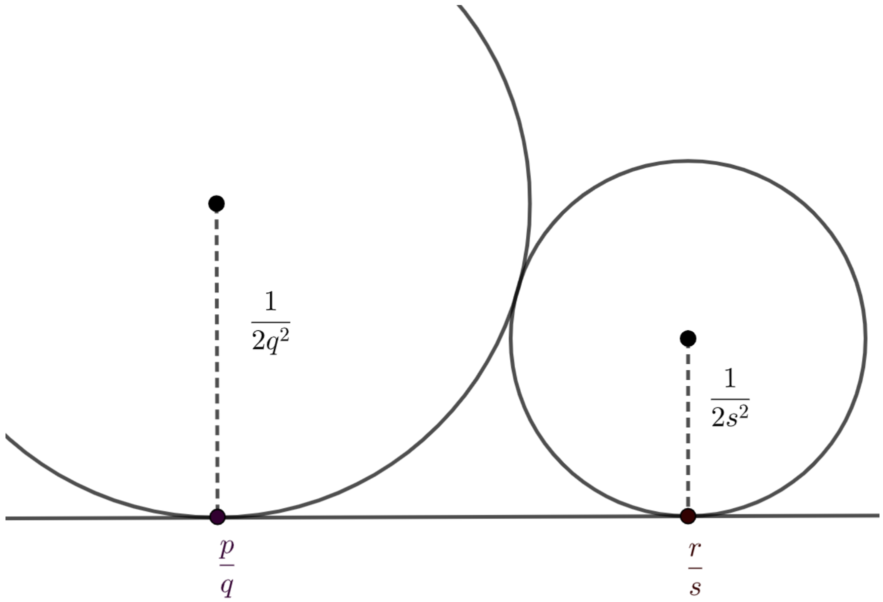
3. The Cookie Monster has n cookies numbered $1, 2 \ldots, n$ arranged in a circle. He goes around the circle starting at cookie number 2, eating every other cookie until all the cookies are gone.

   Let $L(n)$ be the number of the *last* cookie he eats. For example, $L(5) = 3$ because the cookies are eaten in the order $2, 4, 1, 5, 3$.

   (a) Find the following values: $L(16), L(32), L(64)$.

   (b) Let $k > 2$. What is $L(2^k)$ in terms of k?

   (c) Let $n = 2^0 + 2^1 + 2^2 + \cdots + 2^k$. What is $L(n)$ in terms of k?



4. Let p/q be an irreducible fraction, so that $p, q > 0$ are integers with no common factors. Define $C(p, q)$ to be the circle above and tangent to the x-axis, with radius equal to $\frac{1}{2q^2}$.

   (a) Suppose p/q and r/s are irreducible fractions. Prove that $C(p, q)$ and $C(r, s)$ are tangent *if and only if* $|qr - ps| = 1$.

   (b) Suppose $C(p, q)$ and $C(r, s)$ are tangent. Prove that $C(p+r, q+s)$ is tangent to both $C(p, q)$ and $C(r, s)$.