TEMECULA VALLEY
Unified School District

2022 TVUSD

High School Math Competition

Multiple Choice Test (75 minutes)
February 12, 2022

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the scantron with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive +4 points for each correct answer, −1 point for each problem answered incorrectly, and +0 points for each answer left blank.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have 75 minutes to complete the test.

1. Emmy N. is thinking of a square number that can be written in the form $1+p+p^2+p^3+p^4$ for some prime p. It can also be expressed as $k!+1$ for some positive integer k. What is the sum of the digits of Emmy's number?

   (A) 2 (B) 3 (C) 4 (D) 6 (E) 9

2. Leonhard E. wrote letters to 5 of his friends. The post office mixed up the 5 letters and sent each of Leonhard's friends a random letter. What is the probability that at least one friend gets his own letter?

   (A) $\frac{1}{5}$ (B) $\frac{4}{5}$ (C) $\frac{13}{30}$ (D) $\frac{11}{30}$ (E) $\frac{19}{30}$

3. Suppose n is a positive integer so that 2^n and 5^n start with the same digit. What is this digit?

   (A) 1 (B) 3 (C) 4 (D) 7 (E) 8

4. Suppose a, b, and c are integers that satisfy

   $$(x-a)(x-10)+1=(x+b)(x+c) \quad \text{for all } x.$$

   What is the maximum value of $a+b+c$?

   (A) -10 (B) 10 (C) -12 (D) 12 (E) 8

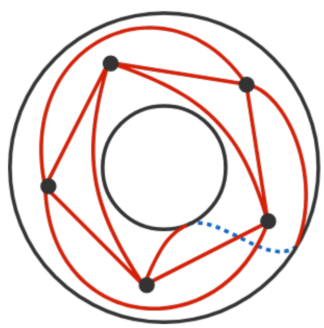


Figure 1: Top view of a graph on a torus

5. A *graph* is a collection of vertices, with some pairs of those vertices connected by edges. For a graph in the plane with no overlapping edges, the graph divides the plane into regions called *faces*. For such a graph, Euler proved $V-E+F=2$ where V, E, and F are the number of vertices, edges, and faces, respectively.
   For a graph on a torus (the surface of a donut) with no overlapping edges, $V-E+F$ is equal to a different constant (see figure above). What is it?

   (A) -2 (B) -1 (C) 0 (D) 1 (E) 3

6. $(1+\tan(1^\circ))(1+\tan(2^\circ))\ldots(1+\tan(45^\circ))$ is an integer. What is its largest prime factor?

   (A) 13 (B) 11 (C) 7 (D) 3 (E) 2

7. When Professor Lovelace divides her class into groups of 3, there is 1 student left out. When she divides them into groups of 5, there are 3 left out. When she divides them into groups of 7,there are 2 left out. Let n be the least possible number of students. What is the sum of the digits of n?

   (A) 13 (B) 19 (C) 10 (D) 9 (E) 2

8. Two acute angles α and β satisfy $\sin^2\alpha + \sin^2\beta = \sin(\alpha+\beta)$. What is $\alpha+\beta$ (in radians)?

   (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$ (E) $\frac{2\pi}{3}$

9. Recall a median of a triangle is a line segment connecting a vertex to the midpoint of the opposite edge. The medians of a triangle T can be rearranged to form a new triangle T'.
   What is the ratio $\frac{\text{Area } T'}{\text{Area } T}$?

   (A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{3}{4}$ (D) $\frac{3}{2}$ (E) 1

10. Define an operation $\bowtie$ on integers by $a \bowtie 0 = a$ for all integers a, and

    $$(a+1) \bowtie b + a \bowtie (b+1) = 3(a \bowtie b) - ab + 2b, \quad \text{for all integers } a, b.$$

    What is $4 \bowtie 672$?

    (A) 2692 (B) 2016 (C) 2684 (D) 1010 (E) 2020

11. The graphs of the two equations

    $$x^2 = y(x+2) + 4$$
    $$16 = (x+2)^2 + y^2$$

    divide the plane into various regions. Find the area of the second smallest region.

    (A) 8 (B) $12\pi + 8$ (C) $4\pi + 8$ (D) $4\pi - 8$ (E) $8\pi + 4$

12. 42 distinct points are chosen in the interior of a regular 2020-gon. Cut the polygon into non-intersecting triangles whose vertices are these 42 points and the 2020 vertices of the polygon. How many triangles is the polygon cut into?

    (A) 2017 (B) 2102 (C) 2104 (D) 2020 (E) 2062

13. For how many integers $1 \le n \le 2020$ is

    $$\frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3}$$

    divisible by 8?

    (A) 1262 (B) 1260 (C) 1332 (D) 1340 (E) 1344

14. Find the sum of all real solutions of the equation

    $$\sqrt[4]{42-x} + \sqrt[4]{x} = 4.$$

    (A) $42 + 2\sqrt{159}$ (B) $2\sqrt{159}$ (C) 21 (D) 42 (E) 46

15. The continued fraction

$$3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cdots}}}$$

can be written as $\dfrac{\sqrt{a}+b}{c}$ where a, b, and c are positive integers and a is prime. What is $a+b+c$?

(A) 5 (B) 10 (C) 16 (D) 18 (E) 19

16. Three circles, with radii of $1, 1$, and 2, are externally tangent to each other. The minimum possible area of a quadrilateral that contains and is tangent to all three circles can be written as $a+b\sqrt{c}$ where c is not divisible by any perfect square larger than 1. Find $a+b+c$.

(A) 32 (B) 22 (C) 20 (D) 19 (E) 18

17. A Mysterious Machine takes as input two positive integers a and b and outputs a number according to the following instructions:

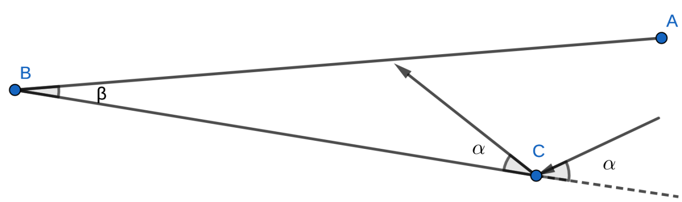
Let $a_0 = a$, $b_0 = b$, and recursively define $a_n = \left\lfloor \frac{a_{n-1}}{2} \right\rfloor$ and $b_n = 2b_{n-1}$ for $n \geq 1$. Stop after N steps when $a_N = 1$. Output the sum of all b_n such that a_n is odd, $0 \leq n \leq N$.

Here $\lfloor x \rfloor$ is the greatest integer less than x. For how many different ordered pairs of postive integers (a, b) will the Mysterious Machine output 2020?

(A) 1 (B) 5 (C) 6 (D) 14 (E) 12

18. A *box* is the finite area in space bound by three pairs of parallel planes. Consider four points in space, not all contained in a single plane. How many boxes have these points as vertices?

(A) 4 (B) 9 (C) 18 (D) 24 (E) 29



19. A light ray is reflected at point C at an angle of incidence $\alpha = 20.02°$, as in the diagram above. If $\angle ABC = \beta = 2.020°$, how many times does the light ray bounce off the segments AB and BC, including the reflection at C?

(A) 69 (B) 70 (C) 71 (D) 35 (E)36

20. Let $f(x) = \dfrac{x\sqrt{3}-1}{x+\sqrt{3}}$ and denote the n-th iterate of f by $f^n(x) = \underbrace{f \circ f \circ \cdots \circ}_{n \text{ times}} f(x)$. What is $f^{2020}(\sqrt{3})$?

(A) $2\sqrt{3}$ (B) $-\sqrt{3}$ (C) $\sqrt{3}$ (D) -1 (E) 1

21. Define an integer-valued function by $f(1) = 1, f(3) = 3$ and

$$\begin{aligned} f(2n) &= f(n), \\ f(4n+1) &= 2f(2n+1) - f(n), \\ f(4n+3) &= 3f(2n+1) - 2f(n). \end{aligned}$$

How many integers $1 \leq n \leq 2020$ satisfy $f(n) = n$?

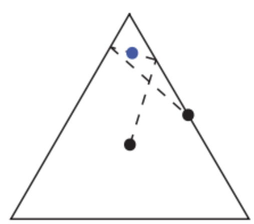
(A) 91 (B) 92 (C) 93 (D) 98 (E) 102

22. A sequence is defined randomly as follows: $a_1 = 5$ or $a_1 = 7$ with equal probability, and $a_{n+1} = 5^{a_n}$ or $a_{n+1} = 5^{a_n}$ with equal probability for $n \geq 1$. What is the sum of all possible values of the last two digits of a_{2020}?

(A) 75 (B) 68 (C) 32 (D) 43 (E) 25

23. A square is inscribed in an ellipse such that two sides of the square respectively pass through the two foci of the ellipse. The square has a side length of 4. The square of the length of the minor axis of the ellipse can be written in the form $a + b\sqrt{c}$ with a, b, and c integers, and c is not divisible by the square of any prime. What is $a + b + c$?

(A) 21 (B) 24 (C) 26 (D) 15 (E) 17



24. A ray of light is emitted in a random direction from the center of an equilateral triangle, whose sides are mirrors. If the light eventually hits the midpoint of a side of the triangle, the midpoint of the light ray path is recorded.
Let S be the set of all possible points recorded in this experiment. How many points are in S?

(A) 16 (B) 12 (C) 10 (D) 11 (E) infinitely many

25. $\triangle ABC$ has side lengths $4, 13$ and 15. Let P be a point inside $\triangle ABC$ with distance x, y, and z from sides AB, BC, and CA, respectively. The minimum value of $x^2 + y^2 + z^2$ can be written as $\dfrac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

(A) 493 (B) 781 (C) 1357 (D) 1449 (E) 2714