# 2020 Temecula Valley
# High School Math Competition

Free Response Test
January 25, 2020

**Name:** ______________________________________________

**School:** ________________________ **ID:** ____________________

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. Print your name, school, and school ID number in the spaces provided above.
3. This section consists of 4 problems, each worth 25 points. These problems are “essay” style questions. You should put all work towards a solution in the space following the problem statement. If you use extra sheets of paper, write your name and the problem number at the top and attach them to this packet.
4. SCORING: You are graded based on the correctness, completeness, and clarity of your solutions. All arguments must be made with mathematical rigor. Clearly state any theorems that you use. Unjustified answers will not receive points.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. When your proctor gives the signal, begin working on the problems. You will have 75 minutes to complete the test.

1. Consider a sphere of radius $r$. A *great circle* is a circle on the surface of the sphere with radius $r$. A *lune* is the area on the surface of the sphere bound between two half great circles that meet at antipodal points (the shaded part in the figure).

   (a) Express the area of a lune with internal angle $\theta$ in terms of $r$ and $\theta$.

   (b) A spherical triangle is the area on the surface of a sphere bound by three arcs of great circles that intersect pairwise (so it has 3 sides and 3 vertices). Let $\alpha, \beta$ and $\gamma$ be the internal angles of a spherical triangle. Express the area of the spherical triangle in terms of $r, \alpha, \beta$, and $\gamma$.

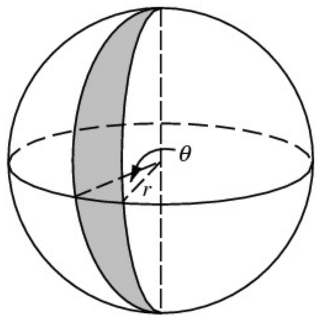


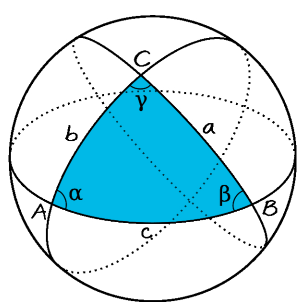

Figure 1: A lune with internal angle $\theta$

*Proof.* (a) The lune is a $\dfrac{\theta}{2\pi}$ fraction of the whole surface area, which is $4\pi r^2$. Hence the area of the lune is $2\theta r^2$.

(b) Denote the spherical triangle $\triangle ABC$. Consider the 3 lunes $\ell_1, \ell_2, \ell_3$ with internal angles $\alpha$, $\beta$, and $\gamma$, with vertices $A, B$, and $C$, resp. Let $A', B'$, and $C'$ be the antipodal points of $A, B$, and $C$, respectively. $\triangle A'B'C'$ is congruent to $\triangle ABC$ and the corresponding lunes $\ell_1', \ell_2', \ell_3'$ with vertices at $A', B'$, and $C'$ are congruent to the lunes with vertices at $A, B$, and $C$. These 6 lunes cover the whole sphere, so counting overlaps we get

$$\begin{aligned}4\pi r^2 &= |\ell_1| + |\ell_2| + |\ell_2| - 2|\triangle ABC| + |\ell_1'| + |\ell_2'| + |\ell_3'| - 2|\triangle ABC| \\ &= 4r^2(\alpha + \beta + \gamma) - 4|\triangle ABC|.\end{aligned}$$

Hence $|\triangle ABC| = r^2(\alpha + \beta + \gamma - \pi)$.




□

**Grading.** *(a) – 2 pts for correct formula. Little justification is needed.*
*(b) – 8 pts*

2. For $n \geq 1$, let

$$M_n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{n \text{ times}}.$$

Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$.

(a) The trace of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined to be $a + d$. Let $L_0 = 2$ and $L_n = \operatorname{trace} M_n$ for $n \geq 1$. Calculate $L_n$ for $n = 1, 2, 3, 4, 5$.

(b) Define $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Prove $L_n = F_{n-1} + F_{n+1}$ for $n \geq 1$.

(c) Find the limit

$$\lim_{n\to\infty} \frac{L_n}{F_n}$$

*Proof.* (a) $1, 3, 4, 7, 11$

(b) We claim that

$$M_n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \qquad \text{for all } n \geq 1.$$

The $n = 1$ case is true. For $n > 1$,

$$M_{n+1} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{n+1}+F_n & F_{n+1} \\ F_n + F_{n-1} & F_n \end{pmatrix}$$
$$= \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}.$$

The claim is proved by induction. Therefore $\operatorname{trace} M_n = F_{n+1} + F_{n-1}$.

(c) Divide the recurrence relation for $F_n$ by $F_{n+1}$ to get

$$\lim_{n\to\infty} \frac{F_{n+2}}{F_{n+1}} = \lim_{n\to\infty} 1 + \frac{F_n}{F_{n+1}}.$$

Assuming the limits exist[1] and equals $L$, the equation becomes $L = 1 + 1/L$. The admissible solution is the positive one, $L = \varphi := \dfrac{1+\sqrt{5}}{2}$. Divide the equation from (b) by $F_n$ and take the limit as $n \to \infty$ to get

$$\lim_{n\to\infty} \frac{L_n}{F_n} = \frac{1}{\varphi} + \varphi = \sqrt{5}.$$

□

**Grading.** *(a) – 1 pt for correct values.*
*(b) – 4 pts. Formal induction is not necessary.*
*(c) – 5 pts. Proof that the limit exists is not necessary. Proof by citing other identities (Binet's formula, Cassini's identity) is also possible.*

[1] To prove it exists, note that the sequence $\{F_{n+1}/F_n\}_{n\geq 0}$ is bounded above by 2 and monotonically increasing.

3. On Mars, Martian students learn about $\mathfrak{M}$, the Martian numbers. Let $\alpha, \beta, \gamma \in \mathfrak{M}$ be any Martian numbers. Multiplication of Martian numbers, written as $*$, satisfies the following rules:

   (R1) If $\beta * \alpha = \beta * \gamma$ then $\alpha = \gamma$

   (R2) $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$

   (R3) There is a special Martian number $\mu$ such that $\alpha^3 = \mu * \alpha * \mu$ for all $\alpha$ (here $\alpha^3 = \alpha * \alpha * \alpha$).

   Note that Martian multiplication is <u>not necessarily commutative</u> (do not assume $\alpha * \beta = \beta * \alpha$). Prove the following statements:

   (a) $\alpha^3 = \alpha$ for all $\alpha \in \mathfrak{M}$.

   (b) $e = \mu^2$ is the identity element. That is, $e * \alpha = \alpha * e = \alpha$ for all $\alpha \in \mathfrak{M}$.

   (c) Martian multiplication is commutative. That is $\alpha * \beta = \beta * \alpha$ for all $\alpha, \beta \in \mathfrak{M}$.

*Proof.* (a) Substitute $\mu * \alpha$ for $\alpha$ in (R3) to get

$$(\mu * \alpha) * (\mu * \alpha) * (\mu * \alpha) = \mu^2 * \alpha * \mu.$$

Using (R1) and (R2) we get $\alpha * (\mu * \alpha * \mu) * \alpha = \mu * \alpha * \mu$. Substituting, $\mu * \alpha * \mu = \alpha^3$ we have

$$\alpha * \alpha^3 * \alpha = \alpha^3.$$

Cancelling $\alpha^2$ using (R1) we get $\alpha^3 = \alpha$ for all $\alpha \in \mathfrak{M}$.

(b) Using $\mu^3 = \mu$ we have $\mu^3 * (\alpha * \mu) = \mu * (\alpha * \mu)$ and hence $\mu^2 * (\alpha * \mu) = (\alpha * \mu)$ by (R1) again. Substitute for $\mu * \alpha * \mu$ using (R3) we get $\mu * \alpha^3 = \mu * \alpha$. By ($a$), this reduces to $\mu * \alpha = \mu * \alpha$, so $\mu$ commutes with all Martian numbers. We get

$$\mu^2 * \alpha = \mu * (\alpha * \mu) = \alpha^3 = (\mu * \alpha) * \mu = \alpha * \mu^2.$$

Hence $\mu^2 * \alpha = \alpha^3 = \alpha = \alpha * \mu^2$, which is what we wanted to show.

(c) We can now cancel $\alpha$ in (a) to get $\alpha^2 = e$, where $e = \mu^2$ is the identity. This shows each $\alpha$ has an inverse $\alpha^{-1}$, and $\alpha^{-1} = \alpha$ (this makes $\mathfrak{M}$ a group). So if $\alpha, \beta \in \mathfrak{M}$

$$\alpha * \beta = (\alpha * \beta)^{-1} = \beta^{-1} * \alpha^{-1} = \beta * \alpha.$$

□

**Grading.** *(a) – 3 pts.*
*(b) – 4 pts. Award 2 pts partial credit for proving $\mu$ commutes with all $\alpha \in \mathfrak{M}$.*
*(c) – 3 pts*

*Award no points if proof uses commutativity or other property not derived from (R1)-(R3).*

4. Remarkably, $m^2 + m + 41$ is a prime number for $m = 0, 1, 2, \ldots, 39$. Let $n \geq 2$ be an integer. Show that if $m^2 + m + n$ is prime for all integers $0 \leq m \leq \sqrt{\frac{n}{3}}$, then $m^2 + m + n$ is also prime for all $0 \leq m \leq n - 2$.

*Proof.* Assume for sake of contradiction that there is a smallest integer $m$ with $\sqrt{n/3} < m \leq n - 2$ such that $m^2 + m + n$ is not prime. Let $p$ be its smallest prime divisor. We must have $p \leq 2m$, for otherwise $p > 2m$ implies

$$\begin{aligned} m^2 + m + n &\geq (2m+1)^2 = 3m^2 + m^2 + 3m + m + 1 \\ &\geq n + m^2 + 3m + 1 \\ &> m^2 + m + n \end{aligned}$$

which is absurd. Now we can write $p = m - k$ if $p \leq s$ or $p = m + k + 1$ if $p > s$, for some $0 \leq k < s$. In either case, we have the factorization

$$m^2 + m + n - (k^2 + k + n) = (m - k)(m + k + 1).$$

Since $p$ divides the right hand side as well as $m^2 + m + n$, it follows that $p | (k^2 + k + n)$. Notice that $m + k + 1 < n - 1 + k + 1 < k^2 + k + n$ and $m - k < n - k < k^2 + k + n$ so $p \neq k^2 + k + n$. Hence $k^2 + k + 1$ is not prime and $k < m$, which contradicts our assumption that $m$ is minimal.

Note that when $n = m - 1$, $m^2 + m + n = (m + 1)^2$ so the bound is sharp. □

**Grading.** *10 pts for complete proof.*
*Award maximum 6 pts partial credit for substantial progress on proof by contradiction or otherwise.*