TEMECULA VALLEY
Unified School District

2020 Temecula Valley
High School Math Competition

Free Response Test
January 25, 2020

Name: ____

School: ___________________________ **ID:** ______________________

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. Print your name, school, and school ID number in the spaces provided above.
3. This section consists of 4 problems, each worth 10 points. These problems are "essay" style questions. Put all work towards a solution in the space following the problem statement and the following blank page. If you use extra sheets of paper, write your name and the problem number at the top and attach them to this packet.
4. SCORING: You are graded based on the correctness, completeness, and clarity of your solutions. All arguments must be made with mathematical rigor. Clearly state any theorems that you use. Unjustified answers will not receive points.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. When your proctor gives the signal, begin working on the problems. You will have 75 minutes to complete the test.

1. Consider a sphere of radius r. A *great circle* is a circle on the surface of the sphere with radius r. A *lune* is the area on the surface of the sphere bound between two half great circles that meet at antipodal points (the shaded part in the figure).

   (a) Express the area of a lune with internal angle θ in terms of r and θ.

   (b) A spherical triangle is the area on the surface of a sphere bound by three arcs of great circles that intersect pairwise (so it has 3 sides and 3 vertices). Let α, β and γ be the internal angles of a spherical triangle. Express the area of the spherical triangle in terms of r, α, β, and γ. (*Hint: use lunes*)

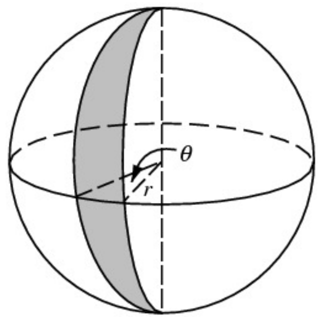

Figure 1: A lune with internal angle θ

2. For $n \geq 1$, let

$$M_n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{n \text{ times}}.$$

Matrix multiplication is defined as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$.

(a) The trace of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined to be $a + d$. Let $L_0 = 2$ and $L_n = \operatorname{trace} M_n$ for $n \geq 1$. Calculate L_n for $n = 1, 2, 3, 4, 5$.

(b) Define $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Prove $L_n = F_{n-1} + F_{n+1}$ for $n \geq 1$.

(c) Find the limit

$$\lim_{n \to \infty} \frac{L_n}{F_n}.$$

3. On Mars, Martian students learn about $\mathfrak{M}$, the Martian numbers. Let $\alpha, \beta, \gamma \in \mathfrak{M}$ be any Martian numbers. Multiplication of Martian numbers, written as $*$, satisfies the following rules:

   (R1) If $\beta * \alpha = \beta * \gamma$ then $\alpha = \gamma$

   (R2) $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$

   (R3) There is a special Martian number μ such that $\alpha^3 = \mu * \alpha * \mu$ for all α (where $\alpha^3 = \alpha * \alpha * \alpha$).

   Note that Martian multiplication is <u>not necessarily commutative</u> (do not assume $\alpha * \beta = \beta * \alpha$). Prove the following statements:

   (a) $\alpha^3 = \alpha$ for all $\alpha \in \mathfrak{M}$.

   (b) μ^2 is the identity element. That is, $\mu^2 * \alpha = \alpha * \mu^2 = \alpha$ for all $\alpha \in \mathfrak{M}$.

   (c) Martian multiplication is commutative. That is $\alpha * \beta = \beta * \alpha$ for all $\alpha, \beta \in \mathfrak{M}$.

4. Remarkably, $m^2 + m + 41$ is a prime number for $m = 0, 1, 2, \ldots, 39$. Let $n \geq 2$ be an integer. Show that if $m^2 + m + n$ is prime for all integers $0 \leq m \leq \sqrt{\dfrac{n}{3}}$, then $m^2 + m + n$ is also prime for all $0 \leq m \leq n - 2$.