TEMECULA VALLEY
Unified School District

2019 TVUSD
High School Math Competition

Multiple Choice Test (75 minutes)
February 2, 2019

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the scantron with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive $+4$ points for each correct answer, -1 point for each problem answered incorrectly, and $+0$ points for each answer left blank.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have 75 minutes to complete the test.

1. Define the operation $*$ by $x * y = x^3 - y$. What is $x * (x * x)$?
   (A) x^3 (B) x (C) $2x$ (D) $-2x$ (E) 0

2. If x and y are nonzero real numbers and $\dfrac{x^2 + y^2}{xy} = 2018$, what is the value of $\dfrac{(x+y)^2}{x^2+y^2}$?
   (A) 1 (B) $\dfrac{2018}{2017}$ (C) $\dfrac{1010}{1009}$ (D) 2018 (E) 2019

3. What is the 2019th term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6 \ldots$?
   (A) 60 (B) 61 (C) 62 (D) 63 (E) 64

4. 84 people are seated in a circle around a very large dining table. Each person randomly spills their drink on either the person to their left or their right. What is the expected number of people who didn't get a drink spilled on them?
   (A) 21 (B) 42 (C) $\dfrac{21}{2}$ (D) 2 (E) 0

5. The roots of the quadratic $x^2 - 163x + k$ are both prime numbers. How many possible values of k are there?
   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

6. Pierre F. is taking the TVUSD Math Competition multiple choice and eliminates one answer choice from each question. He then chooses an answer randomly among the remaining choices. Assuming the correct answers are randomly distributed, what is the expected value of his score?
   (A)$\dfrac{25}{4}$ (B)$\dfrac{25}{2}$ (C) 5 (D) 1 (E) 0

7. In poker, a *pair* is two cards of the same rank and a *three of a kind* is three cards of the same rank. A *full house* consists of 5 cards: a three of a kind and a pair of a different rank, for example $2\diamondsuit, 2\spadesuit, 2\clubsuit, K\clubsuit, K\diamondsuit$. In how many different ways can a full house be made from a standard 52 card deck?
   (A) 624 (B)936 (C) 3744 (D) 5616 (E) 2808

8. Let k be the number of positive 4-digit integers $n \neq 2019$ such that n contains each of the digits 2, 0, 1, and 9 exactly once and $\gcd(2019, n) \neq 1$. What is the remainder when k is divided by 5?
   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

9. The faces of a cube are colored either yellow or orange randomly, independently, and with equal probability. What is the probability that the cube can be placed on a horizontal surface so that the four vertical faces are all the same color?
   (A) $\dfrac{1}{4}$ (B) $\dfrac{5}{16}$ (C) $\dfrac{3}{8}$ (D) $\dfrac{7}{16}$ (E) $\dfrac{1}{2}$

10. Define $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{F_{n-1}F_{n+1}}$$

(A) 2019 (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) 1 (E) Does not converge

11. For which integer k does $x^2 - x + k$ divide $x^{13} + x + 90$?

(A) -5 (B) 5 (C) 6 (D) 3 (E) 2

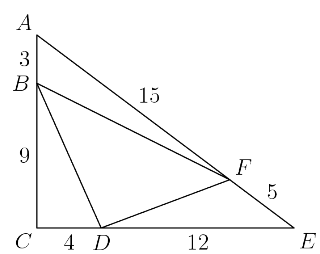
12. Carl G. gets bored in class and multiplies all the even integers greater than 1 and less than 99, except the ones ending in 0. What is the units digit of the product?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

13. A rectangle is inscribed in a circle that has a radius of 1 and an area that is 336 times larger than the area of the rectangle. What is the perimeter of the rectangle?

(A) $\sqrt{4+\frac{\pi}{84}}$ (B) $\sqrt{4+\frac{\pi}{504}}$ (C) $\sqrt{4+\frac{\pi}{42}}$ (D) $\sqrt{16+\frac{\pi}{84}}$ (E) $\sqrt{16+\frac{\pi}{42}}$

14. In the diagram below, $\angle ACE = 90°$, $AC = 12$, and $CE = 16$. B,D, and E are chosen so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of triangle BDF to the area of triangle ACE?

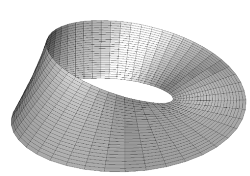


(A) $\frac{7}{16}$ (B) $\frac{9}{25}$ (C) $\frac{3}{8}$ (D) $\frac{11}{25}$ (E) $\frac{1}{4}$

15. What is the smallest positive integer k such that $5^k - 1$ is divisible by 42?

(A) 42 (B) 41 (C) 12 (D) 6 (E) 3

16. A surface is generated by a line segment whose midpoint travels around the unit circle in the xy plane. At each point $(\cos\theta, \sin\theta)$ for $0 \le \theta < 2\pi$ on the unit circle, the line segment makes an angle of $\theta/2$ with the z axis. The generated surface is a Möbius band as shown below. What is the maximum length of the segment generating a surface that does not intersect itself?



(A) $2\sqrt{2}$ (B) $\sqrt{2}$ (C) $\frac{\sqrt{2}}{2}$ (D) $\frac{\sqrt{3}}{2}$ (E) 1

17. A circle is said to *minimize* a set of points if it is a circle with minimal radius such that all the points in the set are inside or on it. 12 points are equally spaced on a circle Γ. A set of 4 out of these 12 points is chosen at random. What is the probability that Γ minimizes these points?

    (A) $\frac{17}{33}$ (B) $\frac{25}{33}$ (C) $\frac{149}{165}$ (D) $\frac{97}{99}$ (E) 1

18. Leonhard E. places k dominoes on a 6×6 grid in an arbitrary configuration, each covering exactly 2 unit squares. He then places another domino on the board without moving any of the other k dominoes. What is the maximum value of k for which this is always possible?

    (A) 13 (B) 12 (C) 11 (D) 10 (E) 9

19. Sophie G. gives you the number $13^4 + 16^5 - 172^2$ and tells you that it is the product of 3 distinct prime numbers. What is the largest of the three prime factors?

    (A) 61 (B) 1321 (C) 1777 (D) 2017 (E) 2207

20. A convex octagon having four consecutive sides of length 2 and four sides of length 3 is inscribed in a circle. The area of the octagon can be expressed as $x + y\sqrt{z}$ with x, y, z positive integers and z prime. What is $x + y + z$?

    (A) 27 (B) 23 (C) 14 (D) 13 (E) 9

21. Emmy N. has a balance and 2019 rocks with unknown weights. She may perform a comparison by placing some of the rocks on one side of the balance and an equal number of rocks on the other side to determine which side weighs more, or if they are equal. What is the minimum number of such comparisons she must perform in order to determine if each individual rock has the same weight?

    (A) 2018 (B) 2019 (C) 2020 (D) 2 (E) 1

22. The equation $x^3 - 10x + 11 = 0$ has three real solutions r,s, and t.
    What is the value of $\arctan r + \arctan s + \arctan t$?

    (A) 1 (B) 0 (C) π (D) $\frac{\pi}{2}$ (E) $\frac{\pi}{4}$

23. Let x, y, z be positive real numbers. Determine the minimum value of $\frac{3x}{y+z} + \frac{4y}{z+x} + \frac{5z}{x+y}$.

    (A)$(\sqrt{3} + \sqrt{4} + \sqrt{5})^2$ (B) $\frac{1}{2}(\sqrt{3} + \sqrt{4} + \sqrt{5})^2$ (C) $(\sqrt{3} + \sqrt{4} + \sqrt{5})^2 - 12$
    (D) $\frac{1}{4}(\sqrt{3} + \sqrt{4} + \sqrt{5})^2 - 12$ (E) $\frac{1}{2}(\sqrt{3} + \sqrt{4} + \sqrt{5})^2 - 12$

24. Three points are chosen randomly on the unit circle, independently and uniformly. Find the probability that the three points lie on a semicircle (a half circle).

    (A) 1 (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{2}{3}$ (E) $\frac{1}{4}$

25. An positive integer k is said to be *tetracomposite* if the number of positive divisors of k (including 1) is equal to the fourth root of k. How many tetracomposite numbers are there?

    (A) 1 (B) 2 (C) 3 (D) 4 (E) Infinitely many