1. On an alien planet, children learn the operation of *μltiplication*, defined by

$$x \otimes y = \frac{xy}{x+y+1}$$

for all x, y non-negative real numbers.

(a) Is $\otimes$ a commutative operation?

(b) Is $\otimes$ an associative operation?

(c) Does $\otimes$ have an identity element?
That is, does there exist $e > 0$ such that $x \otimes e = x$ for all $x \geq 0$?

Solution: $\otimes$ is clearly commutative because regular multiplication and addition are. It is not associative because calculation shows

$$x \otimes (y \otimes z) = \frac{xyz(y+z+1)}{x+y+z+xy+yz+xz+1} \neq \frac{xyz(x+y+1)}{x+y+z+xy+yz+xz+1} = (x \otimes y) \otimes z.$$

If an identity element exist it must satisfy $x \otimes e = x$ for all $x \geq 0$, but

$$x \otimes e = \frac{xe}{x+e+1} = x$$

reduces to $xe = x^2 + xe + x$, and hence $0 = x(x+1)$ which is only solvable for $x = 0$ and $x = -1$. Therefore no identity element exists. □

Remark. *It is also acceptable to provide a single counterexample for (b) or (c), such as*
$(1 \otimes 2) \otimes 3 \neq 1 \otimes (2 \otimes 3)$

2. Isaac N. has a pile of 2019 apples. Each minute he chooses a pile with more than 1 apple, eats an apple from this pile, then divides the remaining pile into 2 smaller, not necessarily equal piles.

   (a) Is it possible for Isaac to make every pile have exactly 6 apples in a finite amount of time?

   (b) Suppose Isaac instead started with k apples. For which k is the answer to (a) yes?

Solution: The answer is $\boxed{no}$ for 2019 apples. The key observation is that every time Isaac increases the number of piles by 1, he decreases the number of apples by 1. Therefore $S :=$ #apples + #piles is invariant. Suppose there are n piles each with 6 apples, then $S = n + 6n = 7n$. But we have $S = 2020$ which is not divisible by 7, a contradiction.

By the previous remarks k must be divisible by 7. Isaac can reach this state by repeatedly making piles of 6 apples since he is decreasing the main pile by 7 each minute. □

3. Let $\{\sigma(1), \sigma(2), \dots \sigma(84)\}$ be a permutation of $\{1, 2, \dots, 84\}$ such that

$$|\sigma(1) - 1| = |\sigma(2) - 2| = |\sigma(3) - 3| = \cdots = |\sigma(84) - 84| > 0$$

That is, the quantities $|\sigma(i) - i|$ are positive and equal for all $i = 1, 2, \dots 84$.
Find the number of such permutations.

Solution: We claim that σ is made up of only 2-cycles, that is, $\sigma(i) = j \implies \sigma(j) = i$.

To this end, assume for sake of contradiction that $\sigma(i) = j$ but $\sigma(j) \neq i$ for some $i, j \in \{1, \dots, 84\}$. First suppose $j > i$. The case $i < j$ is handled identically. Let $k = j - i$ so that

$$k = |\sigma(i) - i| = |\sigma(j) - j|$$

Since $\sigma(j) \neq i$ by hypothesis, we must have $\sigma(j) = i + 2k$ i.e. $\sigma(i + k) = i + 2k$. Similarly,

$$k = |\sigma(i + 2k) - (i + 2k)|.$$

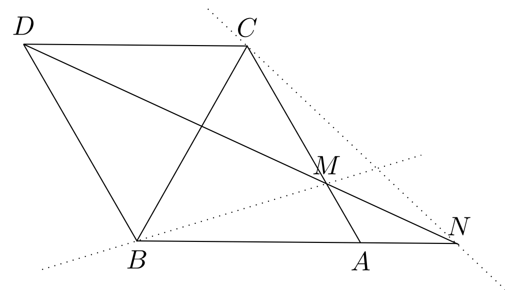
We cannot have $\sigma(i + 2k) = i + k$ since σ is a bijection. Hence $\sigma(i + 2k) = i + 3k$. Continuing this argument we get that $\sigma(i + nk) = i + (n + 1)k$ for each n. However this is impossible since $i + (n+1)k$ is outside the range of σ for some $i + nk \in \{1, \dots, 84\}$. We conclude that σ is made of 2-cycles. In particular, it is an *involution* with no fixed points, meaning $\sigma(\sigma(i)) = i$ and $\sigma(i) \neq i$.

Now suppose we choose k so that

$$\sigma(1) - 1 = \sigma(2) - 2 = \cdots = \sigma(k) - k = k$$

Then we must have $\sigma(k + 1) = 1, \sigma(k + 2) = 2, \dots \sigma(2k) = k$. This choice of k determines σ on the first $2k$ elements of $\{1, \dots, 84\}$. The same argument shows that $\sigma(2k + 1) = 3k + 1, \sigma(2k + 2) = 3k + 2, \dots, \sigma(3k) = 4k$ from which the values of the next $2k$ elements are determined. Thus $\{1, \dots, 84\}$ is partitioned into $2k$ subsets, from which we conclude that k must divide 42.
Therefore there are $\boxed{3}$ such permutations. $\square$

4. ABC and BCD are equilateral triangles sharing the side BC. A line passing through D intersects $\overleftrightarrow{AC}$ at M and $\overleftrightarrow{AB}$ at N. Prove that the acute angle between the lines $\overleftrightarrow{BM}$ and $\overleftrightarrow{CN}$ is $\pi/3$.



Solution: By scaling and rotating we may place the figure in the complex plane so that $B = 0$, $C = 1$, $A = e^{i\pi/3}$, and $D = e^{-i\pi/3}$. Then $N = te^{i\pi/3}$ for some $t \in \mathbb{R}$.

The parametrization of the line ND is

$$z = \lambda \cdot te^{i\pi/3} + (1-\lambda)e^{-i\pi/3}, \qquad \lambda \in \mathbb{R}$$

and the similarly for AC we have

$$z = \gamma e^{i\pi/3} + (1-\gamma), \qquad \gamma \in \mathbb{R}$$

We then calculate M by solving

$$\lambda t \frac{1+i\sqrt{3}}{2} + (1-\lambda)\frac{1-i\sqrt{3}}{2} = \gamma \frac{1+i\sqrt{3}}{2} + (1-\gamma).$$

By equating real and imaginary parts this reduces to solving

$$\begin{aligned} \lambda t + (1-\lambda) &= -\gamma + 2(1-\gamma) \\ \lambda t - (1-\lambda) &= \gamma \end{aligned}$$

from which we obtain $\lambda = 1/t$. Thus M is at $e^{i\pi/3} + (1-1/t)e^{-i\pi/3}$. The desired angle is then the argument of the quotient

$$\frac{e^{i\pi/3} + (1-1/t)e^{-i\pi/3}}{te^{i\pi/3} - 1} = \frac{(e^{i\pi/3} + e^{-i\pi/3}) - (1/t)e^{-i\pi/3}}{te^{i\pi/3} - 1} = \frac{1-(1/t)e^{-i\pi/3}}{te^{i\pi/3} - 1} = \frac{1}{t}e^{-i\pi/3}$$

which is $\pi/3$ as claimed. □

Remark. *A more prescient solution places the coordinates of* A, B, C, D *at* $i\sqrt{3}, -1, 1,$ *and* $-i\sqrt{3}$ *respectively.*