2019 Temecula Valley High School Math Competition

Free Response Test
February 2, 2019

Name: _____

School: _____ **ID:** _____

INSTRUCTIONS

1. DO NOT OPEN TEST BOOKLET UNTIL INSTRUCTED TO DO SO.
2. Print your name, school, and school ID number in the spaces provided above.
3. This section consists of 4 problems, each worth 25 points. These problems are "essay" style questions. You should put all work towards a solution in the space following the problem statement. If you use extra sheets of paper, write your name and the problem number and attach them to this packet.
4. SCORING: You are graded based on the correctness, completeness, and clarity of your solutions. All arguments must be made with mathematical rigor. Clearly state any theorems that you use. Unjustified answers will not receive points.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. When your proctor gives the signal, begin working on the problems. You will have 75 minutes to complete the test.

1. On an alien planet, children learn the operation of *μltiplication*, defined by

$$x \otimes y = \frac{xy}{x+y+1}$$

for all x, y non-negative real numbers.

(a) Is $\otimes$ a commutative operation?

(b) Is $\otimes$ an associative operation?

(c) Does $\otimes$ have an identity element?
That is, does there exist $e > 0$ such that $x \otimes e = x$ for all $x \geq 0$?

2. Isaac N. has a pile of 2019 apples. Each minute he chooses a pile with more than 1 apple, eats an apple from this pile, then divides the remaining pile into 2 smaller, not necessarily equal piles.

   (a) Is it possible for Isaac to make every pile have exactly 6 apples in a finite amount of time?

   (b) Suppose Isaac instead started with k apples. Find all k such that the answer to part (a) is yes.

3. Let $\{\sigma(1), \sigma(2), \ldots \sigma(84)\}$ be a permutation of $\{1, 2, \ldots, 84\}$ such that

$$|\sigma(1) - 1| = |\sigma(2) - 2| = |\sigma(3) - 3| = \cdots = |\sigma(84) - 84| > 0$$

That is, the quantities $|\sigma(i) - i|$ are positive and equal for all $i = 1, 2, \ldots 84$.
Find the number of such permutations.

4. ABC and BCD are equilateral triangles sharing the side BC. A line passing through D intersects $\overleftrightarrow{AC}$ at M and $\overleftrightarrow{AB}$ at N. Prove that the acute angle between the lines $\overleftrightarrow{BM}$ and $\overleftrightarrow{CN}$ is $\pi/3$.